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Exact calculation of the radiatively-induced Lorentz and CPT violation in QED

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Abstract

Radiative corrections arising from the axial coupling of charged fermions to a constant vector b_μ can induce a Lorentz- and CPT-violating Chern-Simons term in the QED action. We calculate the exact one-loop correction to this term keeping the full b_μ dependence, and show that in the physically interesting cases it coincides with the lowest-order result. The effect of regularization and renormalization and the implications of the result are briefly discussed.

The possible breaking of CPT and Lorentz invariance due to non-conventional physics has been recently addressed by constructing extensions of the standard model that include tiny non-invariant renormalizable terms (see Ref.[1, 2, 3] and references therein). In particular, it is interesting to consider the QED sector of such extensions. We shall only be concerned here with the Lorentz-violating CPT-odd terms, which for a single charged (Dirac) fermion read

$$S^{\text{CPT}} = \int d^4x \left[-a_\mu \bar{\psi} \gamma^\mu \psi - b_\mu \bar{\psi} \gamma_5 \gamma^\mu \psi + \frac{1}{2} k^\mu \epsilon_{\mu\nu\rho\sigma} A^\nu F^{\rho\sigma} \right]. \quad (1)$$

Stringent experimental bounds can be put on the pure-photon CPT-violating term [4], which is of the Chern-Simons form [5] (a disputed claim exists, however, for a nonzero \vec{k} [6, 7]). Moreover, this term can generate a negative contribution to the energy that can introduce an instability in the theory [4]. Hence, both experiment and theory suggest that k_μ should vanish. A natural question is then whether a non-zero k_μ can be induced by radiative corrections involving Lorentz and CPT-violating couplings in other sectors of the total low-energy theory. In that case, the tight bounds on k_μ would also constrain these sectors. In the QED extension such corrections can only arise from the axial-vector term, with coupling b_μ .

Several authors have tried to answer this question. All calculations have been performed to one loop and at leading order in b_μ , and have rendered a finite result. However, despite some early claims of definite values for the induced k_μ [2, 8], it seems quite clear now that the result is ambiguous [1, 9, 10, 11], *i.e.*, depends on the details of the high-energy theory [12]. It is our purpose here to calculate the one-loop corrections to all orders in the coupling b_μ ¹ and discuss the relevant issues in the light of the exact result.

The relevant quantity is the parity-odd part of the vacuum polarization, which must be of the form

$$\Pi_{\text{odd}}^{\mu\nu}(p) = \epsilon^{\mu\nu\alpha\beta} b_\alpha p_\beta K(p, b, m), \quad (2)$$

where p_μ is the external momentum and the function K is a scalar. The contribution to the induced Chern-Simons term in the effective action is given by

$$(\Delta k)^\mu = \frac{1}{2} b^\mu K(0, b, m), \quad (3)$$

and must be a function of b^2/m^2 . To one-loop, the only contributing diagram coincides with the standard one-loop vacuum-polarization but with the usual fermion propagator replaced by the b_μ -exact propagator

$$S_b(k) = \frac{i}{\not{k} - m - \not{b}\gamma_5}. \quad (4)$$

We use a hermitian γ_5 with $\text{tr}\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5\} = 4i\epsilon^{\mu\nu\rho\sigma}$ and $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. In order to keep the full dependence on b_μ we must rationalize this propagator. We find

$$S_b(k) = i \frac{(\not{k} + m - \not{b}\gamma_5)(k^2 - m^2 - b^2 + [\not{k}, \not{b}]\gamma_5)}{(k^2 - m^2 - b^2)^2 + 4(k^2 b^2 - (k \cdot b)^2)}. \quad (5)$$

¹When this work was being completed a paper by C.D. Fosco and J.C. Le Guillou pursuing the same aim appeared [13]. However, their result disagrees with the one we present here, except in the massless-fermion case. In their calculation of the massive case, logarithmic divergences appear which are responsible for non-analytic corrections to the effective action. Such corrections would invalidate previous (perturbative) calculations. In fact, there seems to be an algebraic mistake in their derivation of the exact rationalized propagator, for it does not give unity when multiplied by the kinetic term. Nonetheless, their non-perturbative calculation of the radiative corrections in the massless case is correct, and we actually use it as a confirmation of our corresponding result.

Note that if b_μ is space-like and $|b^2| > m^2$, this propagator has not the pole structure of usual propagators: the poles are not on the k_0 real axis but lie on the k_0 imaginary axis. The one-loop vacuum polarization reads

$$\Pi^{\mu\nu}(p) = \int \frac{d^4k}{(2\pi)^4} \text{tr} \{ \gamma^\mu S_b(k) \gamma^\nu S_b(k-p) \}. \quad (6)$$

This integral seems to be linearly divergent, but, as we shall see, the divergent terms cancel. Eq. 5 allows us to compute the trace in the numerator, which for the odd terms in b_μ reduces to $\epsilon^{\mu\nu\alpha\beta} p_\beta F_\alpha(k, p, b, m)$, with

$$\begin{aligned} F_\alpha(k, p, b, m) &= b_\alpha F_\alpha^{(1)}(k, p, b, m) + k_\alpha F_\alpha^{(2)}(k, p, b, m) \\ &= -4i \left\{ b_\alpha \left[2m^2(k^2 - m^2 - b^2) + (k^2 + m^2 + b^2)((k-p)^2 - m^2 + b^2) + 4k \cdot b(k-p) \cdot b \right] \right. \\ &\quad \left. - 2k_\alpha \left[(k^2 - m^2 + b^2)(k-p) \cdot b + ((k-p)^2 - m^2 + b^2)k \cdot b \right] \right\}. \end{aligned} \quad (7)$$

The linearly divergent term has disappeared, leaving an integral which is just logarithmically divergent by power counting. Since we are only interested in $K(p, b, m)$ for $p_\mu = 0$ and, luckily, no term with $\epsilon^{\mu\nu\alpha\beta} b_\alpha k_\beta$ appears, we can simplify the calculation by setting $p_\mu = 0$ in $F_\alpha(k, p, b, m)$ and in the denominator of the integral. We are then left with two integrals which only depend on b_μ and m . The first one is already of the form $b_\alpha K_1(0, b, m)$. The second integral must also give a result of the form $b_\alpha K_2(0, b, m)$, and $K_2(0, b, m)$ can be calculated multiplying the integral by b_α and dividing by b^2 . In this way, we arrive at the following expression:

$$\begin{aligned} K(0, b, m) &= -4i \int \frac{d^4k}{(2\pi)^4} \frac{1}{((k^2 - m^2 - b^2)^2 + 4(k^2 b^2 - (k \cdot b)^2))^2} \\ &\quad \left\{ \left[2m^2(k^2 - m^2 - b^2) + (k^2 + b^2)^2 + 4(k \cdot b)^2 - m^4 \right] \right. \\ &\quad \left. - \frac{1}{b^2} \left[4(k \cdot b)^2(k^2 - m^2 + b^2) \right] \right\}. \end{aligned} \quad (8)$$

In order to calculate this integral using spherical coordinates, we go to Euclidean space via a Wick rotation of the time component of both k_μ and b_μ and get

$$\begin{aligned} K(0, b_E, m) &= 4 \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{((k_E^2 + m^2 - b_E^2)^2 + 4(k_E^2 b_E^2 - (k_E \cdot b_E)^2))^2} \\ &\quad \left\{ \left[2m^2(k_E^2 + m^2 - b_E^2) - (k_E^2 + b_E^2)^2 - 4(k_E \cdot b_E)^2 + m^4 \right] \right. \\ &\quad \left. + \frac{1}{b_E^2} \left[4(k_E \cdot b_E)^2(k_E^2 + m^2 + b_E^2) \right] \right\}, \end{aligned} \quad (9)$$

where we have $b_E^2 = -b^2$. One can directly see at this stage that the result must be finite. Indeed, for very large k^2 the leading term in the integrand has the form

$$\frac{k_E^4 - 4k_E^2(k_E \cdot b_E)^2}{k_E^8}, \quad (10)$$

which gives a vanishing result if the integral is done symmetrically. The other terms are power-counting finite. This also shows an ambiguity in the induced term: the integral of the expression (10) is regularization dependent! We shall further discuss this issue later on. Now, as promised, let us write Eq. (9) in (four-dimensional) spherical coordinates:

$$K(0, b_E, m) = \frac{1}{\pi^3} \int_0^\infty d|k_E| |k_E|^3 \int_0^\pi d\theta \sin^2 \theta \frac{1}{((|k_E|^2 + m^2 - b_E^2)^2 + 4|k_E|^2 b_E^2 \sin^2 \theta)^2} \\ \left\{ \left[2m^2(|k_E|^2 + m^2 - b_E^2) - (|k_E|^2 + b_E^2)^2 - 4|k_E|^2 b_E^2 \cos^2 \theta + m^4 \right] \right. \\ \left. + \left[4|k_E|^2(|k_E|^2 + m^2 + b_E^2) \cos^2 \theta \right] \right\}. \quad (11)$$

This double integral can be carried out in different ways (including numerical integration), which are equivalent as long as spherical symmetry is preserved. The simplest one is to do first the angular integral:

$$K(0, b_E, m) = \frac{1}{\pi^2} \int_0^\infty d|k_E| |k_E|^3 \frac{1}{4b_E^4 |k_E|^2 (|k_E|^2 + m^2 - b_E^2) (4|k_E|^2 b_E^2 + (|k_E|^2 + m^2 - b_E^2)^2)^{3/2}} \\ \left\{ (|k_E|^2 + m^2)(b_E^2 - |k_E|^2 - m^2) \left(4b_E^2 |k_E|^2 + (|k_E|^2 + m^2 - b_E^2)^2 \right)^{3/2} \right. \\ + \text{Sign} \left[|k_E|^2 + m^2 - b_E^2 \right] \left(b_E^8 (m^2 - |k_E|^2) + 6b_E^4 m^2 (|k_E|^2 + m^2)^2 \right. \\ + 2b_E^2 (|k_E|^2 - 2m^2) (|k_E|^2 + m^2)^3 \\ \left. \left. + (|k_E|^2 + m^2)^5 - 2b_E^6 (|k_E|^4 + 3k^2 m^2 + 2m^4) \right) \right\}. \quad (12)$$

This integral is well-behaved for large $|k_E|$. Note the appearance of the sign function, coming from the mentioned pole structure of the propagator. For $m \neq 0$, the final result is (written in terms of the Minkowskian b_μ):

$$\bullet \quad (\Delta k)^\mu = \frac{3}{16\pi^2} b^\mu, \quad \text{if } -b^2 \leq m^2 \quad (13)$$

$$\bullet \quad (\Delta k)^\mu = \left(\frac{3}{16\pi^2} - \frac{1}{4\pi^2} \sqrt{1 - \frac{m^2}{|b^2|}} \right) b^\mu, \quad \text{if } -b^2 > m^2. \quad (14)$$

For any time-like b_μ and for a space-like b_μ with $|b^2| > m^2$, Eq. (13) is the relevant one. Surprisingly enough, in these cases our calculation to all orders in b_μ gives the same result as the one obtained in the b_μ -linear approximation of Ref. [11]. Obviously, perturbation theory about $b_\mu = 0$ does not detect the different behaviour we have found for $-b^2 > m^2$.

The massless-fermion case is more subtle, since the limit $m \rightarrow 0$ does not commute in general with the loop integral. For a space-like b_μ , it can be obtained by taking the appropriate limit of Eq. (14). This gives $(\Delta k)^\mu = -\frac{1}{16\pi^2} b^\mu$. On the other hand, if b_μ is time-like, the integrand in Eq. (11) has a pole at $(k = 1, \theta = \pi/2)$, and one must be careful with the $i\epsilon$ prescription in the Feynman propagator. This pole would not

have appeared if we had kept a non-zero external momentum. Instead of doing that, we provide three other possible ways of calculating the induced term in the massless case, which give the same answer:

1. If one naively ignores the $i\epsilon$ piece, a simpler b_μ -exact propagator can be obtained for $m = 0$:

$$S_b(k) = i \frac{(\not{k} + \not{b}\gamma_5)(k^2 + b^2 - 2p \cdot b\gamma_5)}{(k+b)^2(k-b)^2}. \quad (15)$$

The calculation can then be simplified using Feynman parameters and the result $(\Delta k)^\mu = -\frac{1}{16\pi^2}b^\mu$ is found for any b_μ .

2. Perturbatively, one can perform just a b_μ -linear calculation, since in the massless case the higher-order terms vanish for dimensional reasons. Following the steps of Ref. [11], we see that a shift in one of the linearly divergent integrals gives a $-\frac{1}{16\pi^2}b_\mu$ contribution. The remaining identical integrals vanish in the massless case, up to the ultraviolet ambiguity.
3. A more rigorous (and non-perturbative in b_μ and in the fine-structure constant) confirmation of the same result is provided in Ref. [13] (following a suggestion by D. Colladay further developed in Ref. [9]): an anomalous chiral redefinition of the fermion fields allows to get rid of the coupling to b_μ , so that the contribution (to all orders) to $K(0, b, 0)$ comes from the corresponding Fujikawa Jacobian. Up to the unavoidable ambiguity (which in this method comes from the definition of the current operator), $-\frac{1}{16\pi^2}b^\mu$ is obtained again.

Even in the b_μ -exact result, an infrared regularization can spoil this result. For instance, giving the fermion a small mass obviously shifts it back to $\frac{3}{16\pi^2}b^\mu$ in the time-like- b_μ case. At any rate this is just a formal discussion, since there are no massless electromagnetically-charged fermions in nature.

It is rather striking that the contributions to $(\Delta k)^\mu$ of diagrams with more than one insertion of $\not{b}\gamma_5$ cancel. We have checked that this is indeed the case at order $b_\mu b^2$. At this and higher orders, all integrals are finite by power-counting, and hence unambiguous. After a shift in the loop momentum, one is left with two integrals, corresponding to putting all insertions on the same propagator line, or one on one line and two on the other line. Explicit computation of these integrals shows that, although they do not vanish, they have the opposite sign. Therefore, their sum is zero. We have, however, not found a simple argument for the vanishing of the third-order and higher-order corrections.

Let us discuss now how regularization and renormalization affect the result. This is an important point because the complete $S^{\text{QED}} + S^{\text{CPT}}$ theory is not finite and requires renormalization (and, furthermore, renormalization is also relevant in a finite theory [14, 15]). The *exact* decomposition $S_b(k) = S(k) - iS_b(k)\not{b}\gamma_5 S(k)$ performed in Ref. [11], shows that the ambiguities can only come from the lowest-order piece, the

rest being finite by power counting. Therefore, the ambiguity in the induced Chern-Simons part can be discussed either using the linear approximation, the one-loop exact calculation or the path-integral approach of Ref. [9]. From the discussion above, it is clear that the result of our calculation would be changed by any regularization that destroys the (four-dimensional) spherical symmetry of the high-energy behaviour. This is the case of dimensional regularization, which breaks the (four-dimensional) tracelessness of the combination $k_{\mu\nu} - \frac{1}{4}\delta_{\mu\nu}k^2$. The converse is not true: even if a given regularization preserves spherical symmetry, it may invalidate the steps we followed to arrive at Eq. (9). This happens again in dimensional regularization². In general, different regularizations (or subtractions) will render different results, even if they preserve gauge invariance. This is apparent in differential renormalization, which makes the ambiguity explicit [10] (the situation in the gauge-invariant method of constrained differential renormalization [21] is similar to the one in dimensional regularization [22]).

As a matter of fact, independently of how one regulates and subtracts the divergent integrals, one always has the freedom to add any (renormalizable) finite counterterm that is allowed by the relevant symmetries of the theory [23]. This is also true when the radiative corrections to that term are finite [15]. In the present case, this means that, from the point of view of perturbative renormalization theory, the induced $(\Delta k)^\mu$ can have any value, for it is not protected by any symmetry [11] (except CPT and Lorentz invariance, but we just broke them). The interesting conclusion of our study is the following: if the regulator and the subtraction rule are mass-independent, a mass-independent result will be obtained to all orders in b_μ in the physically-relevant cases, as the CPT- and Lorentz-violating terms coefficients are much smaller than the mass of any electromagnetically-charged fermion and Eq. (13) provides the induced term in this situation.

In Ref. [1] an interesting discussion was made regarding the possible vanishing of the induced Chern-Simons term due to an anomaly-cancellation mechanism in the high-energy theory (of course, we consider now several fermion species). Essentially, the argument goes as follows. From the point of view of a more fundamental theory, the diagrams with one $\not{\gamma}_5$ insertion (at one loop) can be viewed as the corresponding triangular diagrams with the same photon legs and a third leg involving a coupling to an axial vector, in the limit in which there is zero momentum transfer to the axial-vector leg and the latter is replaced with a vacuum expectation value. The condition for the cancelation of the anomalies occurring in these diagrams then implies that the induced term also cancels. This argument requires that the term induced by different fermions

²In the presence of chiral objects, several definitions of dimensional regularization exist. The naive one, usually used in anomaly-free theories, takes an anticommuting γ_5 and is inconsistent with the usual trace relations [16]. It is known to produce a wrong (zero) value for the triangular anomaly, which is closely related to our problem. On the other hand, the consistent definitions of 't Hooft and Veltman [17] give rise to spurious anomalies that must be corrected with finite counterterms [18]. In our calculation we have used an anticommuting γ_5 to invert the kinetic term and then taken usual four-dimensional traces. Finally, dimensional reduction [19] is also ambiguous (even at the one-loop level) in the presence of chiral objects [20]. This short discussion should make clear that one must be careful when applying dimensional regularization to this kind of calculations.

be the same. This is true if the induced term contains no mass dependence and, besides, a universal and mass-independent renormalization prescription is adopted for all the contributing diagrams³. Our result shows that the first requirement, which is trivial at the b_μ -linear order, holds to all orders in b_μ . Invoking the Addler-Bardeen theorem, the argument for the vanishing of the induced term was also generalized in Ref. [1] to higher loops. It must be noted, however, that higher loops can in principle give rise to mass-dependent corrections (even at linear order in b_μ), so that the anomaly-cancellation condition does not forbid the appearance of an induced Chern-Simons term.

On the other hand, we have till now neglected the possible influence of the a_μ term in Eq. (1). In principle, corrections of order $a^2 b_\mu$ and higher could appear, and they are not necessarily smaller than the ones we have been dealing with. Actually, one can include the effect of a_μ to all orders just by considering the corresponding a_μ - and b_μ -exact propagator. This propagator is just the one in Eq. (5) but with k_μ substituted by $k_\mu - a_\mu$. The vector a_μ behaves then like an external momentum which appears in all the propagators of the loop. Since there are no derivative couplings, a shift in the loop momentum $k_\mu \rightarrow k_\mu + a_\mu$ can completely eliminate a_μ , so the result is not affected. (Of course, this shift is also subjected to regularization ambiguities.) The situation is not so simple, however, if the effect of other sectors (like the CPT-even Lorentz-violating extension of QED considered in Ref. [1]) is taken into account. This is beyond the scope of the present work.

Let us finally stress that even if the radiative corrections to the Chern-Simons term cancel for some reason, it is still possible to add a finite counterterm and get a non-zero $(\Delta k)^\mu$. This is a sign of the fact that we have no right to put $k^\mu = 0$ at tree level, unless there is some symmetry in the high-energy theory that imposes this value. In the absence of such a symmetry, we are facing a problem of naturalness. On the other hand, if one could really get $(\Delta k)^\mu = 0$ in a mass-independent way, one would have at least a consistent low-energy theory, which would include the prescription $k^\mu + (\Delta k)^\mu = 0$. The validity of such theory would only be justified *a posteriori* by the mentioned theoretical and experimental reasons.

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³Such a prescription can be justified again by a similar argument.

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